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Endogenous Growth with Addictive Habits*

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Abstract

In this paper, we investigate the global dynamics of an endogenous growth model with linear technology and addictive habits. We find feasible parameters' conditions under which: a) the resulting equilibrium consumption path is steeper than in a standard AK model; b) endogenous fluctuations in the form of damping fluctuations around the balanced growth path emerge; c) the Easterlin's paradox emerges. The relevance of these results is explained comparing our findings with the results already known in the existing literature.

Keywords: Habit formation; consumption smoothing, endogenous fluctuations.

JEL Classification: E00, E21, O40.

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1 Introduction

In this paper we fully characterize the global dynamics of an AK growth model with addictive external habits.¹ Habits formation is described by the following weighted average of past consumption, with the weights decreasing exponentially into the past:

$$h(t) = \varepsilon \int_{t-\tau}^t \bar{c}(u) e^{\eta(u-t)} du$$

where $\varepsilon > 0$, $\eta \geq 0$, and $\tau > 0$ indicate respectively the *intensity*, *persistence* and *lag structure* of the habits, $h(t)$, while $\bar{c}(u)$ is average consumption. This description of habits formation is general relative to the assumptions on their intensity, persistence, and lag structure and it embeds all the main specifications used in the literature as explained in Remark 1 in the next Section. In particular, there is no contribution in the literature, as far as we know, studying the implication of habit formation in an endogenous growth framework *without imposing additional restrictions* on the parameters ε , η , and τ .²

The global dynamics, which may emerge in this model, is quite rich and there are several reasons why it is worth to be investigated.³

First of all, a full characterization of the global dynamics unveils how the (asymptotic) growth rate of the economy is determined. In fact, we show that the growth rate of the economy can be pinned down either by the modified growth rate of the shadow price of capital⁴ - exactly as in the standard AK model - or by the growth rate of the habits. In particular, we prove that if habits take the form of consumption aspirations (i.e. $\varepsilon > \eta$, see also later Definition 1) then they may grow faster than the modified growth rate of the shadow price of capital and can pin down the growth rate of all the other aggregate variables; even more, we show that positive growth is possible even if the modified growth rate of the shadow price of capital is negative. Therefore, the presence of consumption aspirations may imply positive growth under the same conditions on the exogenous parameters which would have implied negative growth in the standard AK

¹Addiction means that consumption has to increase as the habits accumulate; it has been introduced in the model by considering a subtractive nonseparable utility function. Addictive habits are often assumed in the literature (see Boldrin et al. [8], Constantinides [17] among others).

²As it will result clear in the following such restrictions rule out two interesting cases: the possibility of a consumption equilibrium path steeper than its counterpart in a model without habit, and the possibility of endogenous damping fluctuations. Also the Easterlin's paradox cannot be obtained with such restrictions.

³Gomez [23] has recently investigated the global dynamics of a model with external habits but without habit addiction. Also the specification of the habit formation is different since it is assumed (using our notation) that $\varepsilon = \eta$, and $\tau = \infty$. Other relevant contributions which have studied the dynamics of habits formation models without addiction are Carroll et al. [14], [15].

⁴The modified growth rate of the shadow price of capital is defined as the product of the growth rate of the shadow price of capital with the opposite of the elasticity of substitution; in next section will be $\Gamma = -(A - \delta - \rho) \cdot (-\frac{1}{\gamma})$.

model.⁵

The global dynamics of an economy, whose positive growth rate is driven by consumption aspirations, has the following features: firstly, the consumption equilibrium path is steeper than in a model without consumption aspirations, because such growth rate can be achieved, everything else equal, only if the agents save more at the beginning; secondly, the growth rate of output and the growth rate of the instantaneous utility may be different and may even have opposite sign. Therefore it is possible to specify the exogenous parameters to have constant utility over time but positive output growth - i.e. the model can easily replicate the so called Easterlin's paradox (see Blanchflower and Oswald [7] among others). However, it is not possible to use the model to explain this fact and at the same time other empirical evidences, such as the equity premium puzzle, where the key parameters in the habits formation equation has to be specified to make the consumption equilibrium path flatter than in the case without habits (i.e. $\varepsilon \leq \eta$ and then no consumption aspiration - see Constantinides [17] among others).

A full understanding of the global dynamics of this model represents also a contribution to the existing literature on endogenous fluctuations in habits' formation models because we prove that damping fluctuations around the balanced growth path may arise if the following conditions are met: i) the habits take the form of consumption aspirations; ii) the habits are formed over a finite and sufficiently large interval of time (i.e. τ sufficiently large but finite); iii) the modified growth rate of the shadow price of capital is negative. Keeping aside the distinction between internal and external habits, our finding differs from the existing contributions because the raising of endogenous fluctuations is proved in an endogenous growth setting, while previous contributions, such as Ryder and Heal [26] and Benhabib [5], focused on models without growth, and because of the different source of the fluctuations. In fact, aspiration, a finite lag structure, and a negative modified growth rate of the shadow price of capital are, in our paper, the three necessary ingredients for the endogenous fluctuations around the balanced growth path to emerge while, in the two just cited contributions, endogenous fluctuations depend on the value of the preference discount factor. The source of the endogenous fluctuations in our paper is also different from the one described in Dockner and Feichtinger [21] because they proved, in a model with a single consumption good which accumulates two stocks of consumption capital, one generating addictive internal habit and the other one satiating behavior, that the "conflicting role" played by the addictive and satiating behavior over the same good is responsible of the raising of endogenous fluctuations.⁶

Finally our analysis represents also a contribution to the large literature on applica-

⁵Of course, the growth rate of the economy has to be lower, in any case, than the real interest rate, otherwise it is not sustainable. A growth rate lower than the interest rate may emerge under some parameters' restriction because we assume constant return to scale in production which implies a constant and positive interest rate.

⁶The example provided by these authors is indeed very illustrative: "only a consumer with a desire to eat and a dislike for weight who anticipates the future consequences of his current actions can end up in eating and dieting cycles".

tions of delay differential equations to economic problems, a literature which includes, for examples, vintage capital models (see Boucekkine et al. [9], [11], [13], [10], [12] among others), and time-to-build models (see, for example, Asea and Zak [2] and Bambi et al. [3]). In fact, differently from what generally found in these models, the presence of a delay differential equation is not enough by itself to induce endogenous fluctuations since in our model monotonic convergence to the balanced growth path can be found also when the habits formation equation is a delay differential equation.

2 Description of the Economy

Consider a standard neoclassical growth model, where the economy consists of a continuum of identical infinitely lived atomistic households, and firms. The households' objective is to maximize over time the discounted instantaneous utility, $u(c(t), h(t))$, which is a function of current consumption, $c(t)$, and external habits $h(t)$. It is indeed assumed, as in Constantinides [17], and in many others contributions in the macro-finance literature (e.g. Chapman [16], or one of the specifications in Detemple and Zapatero [19]), that the instantaneous utility function has the no-separable subtractive form:

$$u(c(t), h(t)) = \frac{(c(t) - h(t))^{1-\gamma}}{1-\gamma}, \quad (1)$$

for $c(t) \geq h(t)$ and $\gamma > 0$ but different from 1. Observe that if $c(t) < h(t)$ then the utility function is not well defined in the real field for some values of γ (e.g. $\gamma = \frac{3}{2}$), and, it is never concave. For this reason, it is generally assumed that $u(c(t), h(t)) = -\infty$ as soon as $c(t) < h(t)$. The instantaneous utility function (1) implies *addictive habits* because current consumption is forced to remain higher than the external habits over time; alternatively it can be seen as a Stone-Geary utility function with an endogenous and time varying subsistence level of consumption, $h(t)$. Our analysis focuses on an exponentially smoothed index of the economy past average consumption rate as a mechanism of habit formation:

$$h(t) = \varepsilon \int_{t-\tau}^t \bar{c}(u) e^{\eta(u-t)} du \quad \text{or} \quad \dot{h}(t) = \varepsilon (\bar{c}(t) - \bar{c}(t-\tau) e^{-\eta\tau}) - \eta h(t) \quad (2)$$

where $\eta \geq 0$ measures the persistence of habits, $\varepsilon > 0$ the intensity of habits, i.e. the importance of the economy average consumption relative to current consumption, and finally $\tau > 0$ is the lag structure or memory parameter.⁷

Remark 1 *The habit formation equation (2) includes all the specifications of (addictive) habits generally used by the literature; for example:*

⁷The external habits at two sufficiently far dates may be completely unrelated under a finite choice of τ ; in fact, only the average consumption between t and $t-\tau$ matters: in this respect, a finite τ introduces a complete and periodic update of the habit.

- a) If $\varepsilon \leq \eta$ and $\tau \rightarrow +\infty$, equation (2) describes the habits as in Constantinides [17];
- b) If $\varepsilon > \eta = 0$ and $\tau = 1$, equation (2) represents the (continuous-counterpart of the) habits as intended by Boldrin et al. [8].

It is also worth noting that a choice of τ greater than one and lower or equal than three is consistent with Crawford's [18] econometric estimates. The following definition will turn out useful in the remaining of the paper.

Definition 1 (Consumption Aspirations) *Habits evolving as in equation (2) under the parameters' restriction $\varepsilon > \eta$ are called consumption aspirations.*

The representative household solves the following problem where the habits enter as an externality in the instantaneous utility function:

$$\begin{aligned} & \max \int_0^\infty \frac{(c(t) - h(t))^{1-\gamma}}{1-\gamma} e^{-\rho t} dt \\ & s.t. \quad \dot{k}(t) = (R(t) - \delta)k(t) - c(t) \\ & \quad k(t) \geq 0, \quad c(t) \geq h(t) \geq 0 \\ & \quad k(0) = k_0 > 0 \end{aligned}$$

with $R(t)$ the rental rate of capital, and δ the depreciation rate. From now on we will assume a priori that the control is interior and the state constraint is satisfied. Then we will identify a posteriori what are the restrictions to be imposed so that all the inequalities constraints hold. Following this strategy, we may easily write the first order conditions from the present value Hamiltonian

$$c(t) - h(t) = \varphi(t) \tag{3}$$

$$\frac{\dot{\varphi}(t)}{\varphi(t)} = \frac{1}{\gamma}(r(t) - \rho) \tag{4}$$

where $\varphi(t) = \psi(t)^{-\frac{1}{\gamma}}$ with $\psi(t)$ the shadow price of capital which is the costate variable of our problem. Then the representative household's behavior is fully described by these first order conditions, and the transversality condition

$$\lim_{t \rightarrow \infty} \psi(t)k(t) = 0$$

Finally the representative firm maximizes its profit function subject to a linear production function with capital the only factor. Then the rental rate of capital is constant over time, $R(t) = A$ as well as the interest rate $r(t) = R - \delta = A - \delta \equiv r$.

The competitive equilibrium for this economy is defined taking into account the household's intertemporal maximization problem, the habits' formation equation, (2), the firm's static problem, and the market clearing conditions on the capital and final good market. At the equilibrium we have also that $\bar{c}(t) = c(t)$ and that the growth rate of $\varphi(t)$, or modified growth rate of the shadow price of capital, is $\Gamma = \frac{r-\rho}{\gamma}$.

Definition 2 *A competitive equilibrium is any trajectory $\{\varphi(t), k(t), h(t)\}_{t \geq 0}$ which solves*

$$\dot{k}(t) = rk(t) - \varphi_0 e^{\Gamma t} - h(t) \quad (5)$$

$$h(t) = \varepsilon \int_{t-\tau}^t [h(u) + \varphi_0 e^{\Gamma u}] e^{\eta(u-t)} du \quad (6)$$

$$\varphi(t) = \varphi_0 e^{\Gamma t} \quad (7)$$

and respects (i) the initial condition of capital, $k(0) = k_0$; (ii) the habits' past history, $h_0(t)$, $t \in [-\tau, 0]$; (iii) the transversality condition $\lim_{t \rightarrow \infty} k(t)e^{-rt} = 0$ and (iv) the inequality constraints $k(t) \geq 0$ and $\varphi(t) \geq 0$.

Observe that the problem is well posed as long as we specify an initial continuous function, $h_0(t)$ with $t \in [-\tau, 0]$; as a consequence the past history of consumption, namely $c(t) \in [-2\tau, 0)$ is known, but not $c(0) = c_0$ whose value will be determined by (univocally) choosing the initial shadow price of capital, $\psi(0)$, which rules out the transversality condition and pins down the unique equilibrium path of the economy. It is worth noting that the system of equations (5), (6), and (7) has a triangular form and can be solved starting from the last equation.

3 Global dynamics when τ is infinite.

We first characterize the global dynamics in the easier case with τ infinite. Under this assumption, equation (6) can be rewritten as a non-autonomous ordinary differential equation. Its solution and the results on the global dynamics of the economy will be found in the next proposition. Before proceeding, we briefly remind that in a standard AK model, without habits, all the aggregate variables “jump” immediately to the balanced growth path where they grow at the rate $\Gamma = \frac{1}{\gamma}(r - \rho)$; then the economy faces positive growth if and only if the interest rate is sufficiently high:

$$r \geq \rho \quad \Leftrightarrow \quad \Gamma \geq 0 \quad (8)$$

If this condition is not met, the economy shrinks over time.

Proposition 1 *Assume $k_0 \geq h_0 \frac{1}{r-\varepsilon+\eta}$, and $0 < \max(\varepsilon - \eta, \Gamma) < r$.⁸ Then a decentralized economy with external addictive habits, and linear production, has:*

i) a unique positive asymptotic growth rate $g = \max(\varepsilon - \eta, \Gamma)$;⁹

⁸The condition $k_0 > h_0 \frac{1}{r-\varepsilon+\eta}$ guarantees that the interior solution, where $c(t) > h(t)$ for any $t \geq 0$, satisfies the maximum principle. On the other hand, the same condition with equality guarantees that the corner solution, where $c(t) = h(t)$ for any $t \geq 0$, satisfies the maximum principle (e.g. Hartl et al. [24], Theorem 4.2).

⁹The asymptotic growth rate of a variable x is defined as $\lim_{t \rightarrow +\infty} \frac{\dot{x}}{x}$.

ii) a unique balanced growth path

$$k(t) = k_0 e^{gt}, \quad c(t) = c_0 e^{gt}, \quad \text{and} \quad h(t) = h_0 e^{gt}$$

with either $c_0 = h_0 = (r - g)k_0$ if $g = \varepsilon - \eta$ or $c_0 = \frac{g+\eta}{\varepsilon} h_0 = (r - g)k_0$ if $g = \Gamma$;¹⁰

iii) a unique equilibrium path converging monotonically to the balanced growth path:

$$h(t) = \frac{\varepsilon \tilde{\varphi}_0}{\Gamma - \varepsilon + \eta} e^{\Gamma t} + \left(h_0 + \frac{\varepsilon \tilde{\varphi}_0}{\varepsilon - \eta - \Gamma} \right) e^{(\varepsilon - \eta)t} \quad (9)$$

$$c(t) = e^{(\varepsilon - \eta)t} \left(h_0 + \frac{\varepsilon \tilde{\varphi}_0}{\varepsilon - \eta - \Gamma} \right) + \tilde{\varphi}_0 e^{\Gamma t} \left(\frac{\eta + \Gamma}{\Gamma - \varepsilon + \eta} \right) \quad (10)$$

$$k(t) = \left[k_0 - \frac{h_0}{r - \varepsilon + \eta} - \frac{(r + \eta)\varphi_0}{(r - \varepsilon + \eta)(r - \Gamma)} \right] e^{(r)t} + \frac{1}{r - \varepsilon + \eta} \left(h_0 + \frac{\varepsilon \tilde{\varphi}_0}{\varepsilon - \eta - \Gamma} \right) e^{(\varepsilon - \eta)t} - \frac{(\eta + \Gamma)\varphi_0}{(\varepsilon - \eta - \Gamma)(r - \Gamma)} e^{\Gamma t} \quad (11)$$

with

$$\tilde{\varphi}_0 = \frac{(\varepsilon - \eta - A + \delta)(\Gamma - A + \delta)}{(r + \eta)} \left(k_0 + h_0 \frac{1}{\varepsilon - \eta - A + \delta} \right) \quad (12)$$

One of the results of this proposition is that an economy with consumption aspirations and $\varepsilon - \eta > \Gamma$ grows faster than a standard AK economy. Consumption aspirations may indeed imply positive growth even if $\Gamma < 0$, and the standard AK economy would shrink over time. Why this departure from the standard predictions of the AK model? Intuitively this result depends on the combination of consumption aspirations and addiction. In fact, the representative consumer faces two constraints

$$\dot{h}(t) = (\varepsilon - \eta)h(t) + \varepsilon(\bar{c}(t) - h(t)) \quad \text{and} \quad c(t) \geq h(t) \quad (13)$$

At the equilibrium, the solution path of $c(t)$, obtained by solving the representative agent problem, coincides with the given path, $\bar{c}(t)$; when this happens, (13) implies:

$$\dot{h}(t) = (\varepsilon - \eta)h(t) + \varepsilon(c(t) - h(t)) \geq (\varepsilon - \eta)h(t) \quad (14)$$

and the habit stock, as well as aggregate consumption, grows at least at the rate $\varepsilon - \eta$ even if $c(t) - h(t)$ shrinks over time at the rate $\Gamma < 0$. This growth rate is sustainable when lower than the interest rate of the economy, $r = A - \delta$, and unless the initial stock of habit is too high.

The restriction on the initial conditions becomes clearer when $\varepsilon = \eta$; in this case, the constraint becomes $rk_0 \geq h_0$ and implies that the initial habits have to be lower or equal than the household's initial net income otherwise the initial level of consumption can be higher or equal than the initial habits, ($c_0 \geq h_0$), only if an initial disinvestment ($i_0 < 0$) takes place. If this happens a positive growth rate of consumption and habits is clearly

¹⁰Of course, the economy will be on the balanced growth path from $t \geq 0$ only for not generic initial conditions of the two state variables (i.e. this set has zero measure); for generic initial conditions the economy will be always characterized by transitional dynamics as emerge from iii) in this Proposition.

not sustainable over time because financed with a continuous reduction of capital which will eventually converge to zero. This restriction on the initial condition becomes more (less) stringent if $\varepsilon > \eta$ ($\varepsilon < \eta$) since now the habits accumulation is faster (slower) than before.

Before concluding this section three remarks are necessary.

Remark 2 *The utility is always bounded, i.e. $\int_0^\infty u(c(t), h(t))e^{-\rho t} dt < \infty$, as long as $(1 - \gamma)\Gamma - \rho < 0$. This is the same condition as in the standard AK model.*

Remark 3 *The balanced growth path with $g = \varepsilon - \eta$ (see Proposition 1) is a corner solution of the economy since the constraint, $c(t) \geq h(t)$, is binding.*

Remark 4 *The transitional dynamics and balanced growth properties of an AK model with internal and external addictive habits coincide when $g = \Gamma$ (see Gomez [22]).*

4 Global dynamics when τ is finite.

4.1 Habits and consumption dynamics

We now characterize the habits and consumption dynamics in the case $\tau > 0$ and finite. Under this assumption, equation (6) is a non autonomous algebraic equation.¹¹ To rewrite it as an autonomous algebraic equation, we consider the variable change

$$z(t) = h(t) - e^{\Gamma t} \varphi_0 \theta, \quad (15)$$

where $\theta = \frac{\varepsilon \int_{-\tau}^0 e^{(\eta+\Gamma)(u-t)} du}{1 - \varepsilon \int_{-\tau}^0 e^{(\eta+\Gamma)(u-t)} du}$ (computational details are reported in the Appendix, Lemma A), and we rewrite (6) as

$$z(t) = \varepsilon \int_{t-\tau}^t z(u) e^{\eta(u-t)} du \quad (16)$$

Equation (16) is autonomous and then easier to study than equation (6).¹² The characteristic equation of (16) is

$$\Delta(\lambda) = 0 \quad \text{with} \quad \Delta(\lambda) = -1 + \varepsilon \int_{-\tau}^0 e^{(\lambda+\eta)u} du \quad (17)$$

This equation has an *infinite number of complex conjugate complex roots* in each half plane $Re(\lambda) < \lambda_0$, for any $\lambda_0 \in \mathbb{R}$ (e.g. Diekmann et al. [20], Chapter XI). We will be mainly interested in locating the roots with the larger real part. This is done in the following lemma:

¹¹This equation can be rewritten as a non-autonomous linear delay differential equation. Vintage capital models are often characterized by differential equations with delays as documented, for example, by Benhabib and Rustichini [6], and Boucekkine et al. [9], [11], [13].

¹²See also Augeraud-Veron et al.[1] for further discussion on this class of functional differential equations.

Lemma 1 *The spectrum of roots of (17) has*

- *a unique leading real root $\alpha^* > \max(0, \Gamma)$ if and only if*

$$\varepsilon - \eta > \max(0, \Gamma) \quad \text{and} \quad \tau > \tau^* \quad \text{with} \quad \tau^* = -\frac{\log\left(1 - \frac{\eta + \max(0, \Gamma)}{\varepsilon}\right)}{\eta + \max(0, \Gamma)} \quad (\text{Condition 1})$$

- *an infinite number of complex conjugate roots with real part smaller than $\min(-\eta, \alpha^*)$.*

Moreover α^* is an increasing function of τ and converges to $\varepsilon - \eta$, as soon as $\tau \rightarrow \infty$.

Observe that Condition 1 implies households with consumption aspirations and a sufficiently high memory parameter, τ . We may now use this information on the spectrum of roots to find the solution of (16).

Proposition 2 *The solution of the autonomous-algebraic equation (16) is*

$$z(t) = \underbrace{z_0 e^{\alpha^* t}}_{\text{trend component}} + \underbrace{\frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda}_{\text{oscillatory component}} \quad (18)$$

where

$$F(\lambda) = -\varepsilon \Delta(\lambda)^{-1} \left(\int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 (h(\mu) - e^{\Gamma\mu} \varphi_0 \theta) e^{-\lambda\mu} d\mu \right) du \right)$$

with $\gamma > \sup \{ \operatorname{Re}(\lambda) < \alpha^*, \det(\Delta(\lambda)) = 0 \}$ and $L(\gamma) = \{ \lambda \in \mathbb{C}, \operatorname{Re}(\lambda) = \gamma \}$. Finally z_0 is equal to

$$z_0 = -\frac{\varepsilon}{\Delta'(\alpha^*)} \left[\frac{1}{\eta + \alpha^*} \int_{-\tau}^0 h(\mu) e^{\eta\mu} d\mu - \frac{e^{-(\eta+\alpha^*)\tau}}{\eta + \alpha^*} \int_{-\tau}^0 h(\mu) e^{-\alpha^*\mu} d\mu \right. \\ \left. - \frac{\varphi_0 \theta}{(\Gamma - \alpha^*)} \left(\frac{1 - e^{-(\eta+\alpha^*)\tau}}{(\eta + \alpha^*)} - \frac{1 - e^{-(\Gamma+\eta)\tau}}{(\Gamma + \eta)} \right) \right] = z_0(h_0(t), \varphi_0) \quad (19)$$

and $\lim_{t \rightarrow \infty} z(t) e^{-\alpha^* t} = z_0$.

The solution of $z(t)$ has, therefore, two components: a trend component, whose presence depends on the existence of the leading positive real root α^* , and an oscillatory component which is spanned by the infinite conjugate complex roots. This oscillatory component induces oscillations around the trend; these oscillations tend to die out over time since $\lim_{t \rightarrow \infty} z(t) e^{-\alpha^* t} = z_0$. Observe also that z_0 is a function of the initial history of the habits and the initial stock of capital. From the solution of $z(t)$ we may derive the dynamics of consumption and of the habits using the change of variable (15):

Proposition 3 *Consumption and habit evolve over time as described by the following equations:*

$$c(t) = \underbrace{\varphi_0 e^{\Gamma t} (1 + \theta) + z_0 e^{\alpha^* t}}_{\text{trend component}} + \underbrace{\frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda}_{\text{oscillatory component}} \quad (20)$$

$$h(t) = \underbrace{\varphi_0 e^{\Gamma t} \theta + z_0 e^{\alpha^* t}}_{\text{trend component}} + \underbrace{\frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda}_{\text{oscillatory component}} \quad (21)$$

Then the asymptotic positive growth rate of consumption and the habit stock is $g = \alpha^*$ when Condition 1, found in Lemma 1, holds and $z_0 \neq 0$.¹³ It will be $g = \Gamma$ if Condition 1 does not hold.

From Lemma 1 and Proposition 3, it emerges that the introduction of a finite lag structure leads to two changes in the aggregate consumption dynamics. First, under Condition 1, the growth rate of consumption, α^* , depends positively on τ and it will be always lower or equal than in the model with τ infinite; even more a choice of τ sufficiently low violates Condition 1, breaking down the possibility of a higher growth rate than in the standard AK model.

This result can be explained looking at the counterpart of equations (13) and (14) when τ is finite. In particular, we may observe that the representative consumer faces now the two constraints

$$\dot{h}(t) = \varepsilon (\bar{c}(t) - \bar{c}(t - \tau)e^{-\eta\tau}) - \eta h(t) \quad \text{and} \quad c(t) \geq h(t) \quad (22)$$

which, at the equilibrium, lead to the following inequality

$$\dot{h}(t) = \varepsilon (c(t) - c(t - \tau)e^{-\eta\tau}) - \eta h(t) \quad (23)$$

$$\geq (\varepsilon - \eta)h(t) - \varepsilon c(t - \tau)e^{-\eta\tau} \quad (24)$$

Then the presence of a finite τ adds the last component in relation (24) which may reduce or even break down the mechanism for a positive growth rate in the habits and then in the aggregate consumption. The following remark is also useful

Remark 5 *Risk adverse agents choose a smooth path of $c(t) - h(t)$. Consistently with that, the oscillatory components in consumption and habits are identical and thus cancel out.*

Observe also that at this stage φ_0 is not yet specified and then equations (20) and (21) are not the competitive equilibrium paths; they will become competitive equilibrium paths after having determined (if any) the $\varphi_0 = \tilde{\varphi}_0$ which makes the transversality condition hold.

4.2 Dynamics of the competitive equilibrium

In this section we prove the existence and uniqueness of the equilibrium path. The transitional dynamics as well as the asymptotic growth rate of all the aggregate variables will be also found. These results will be proved in Proposition 4 using the solution of consumption and habits obtained in the previous section.

¹³It will be indeed shown in subsection 4.2 that the conditions for an interior solution will also guarantee that $z_0 > 0$.

Proposition 4 Assume $0 < \max(\alpha^*, \Gamma) < r$ and

$$k(0) \geq \frac{h(0)}{r + \eta - \varepsilon + \varepsilon e^{-(r+\eta)\tau}} - \frac{\varepsilon e^{-(r+\eta)\tau}}{r + \eta - \varepsilon + \varepsilon e^{-(r+\eta)\tau}} \int_{-\tau}^0 e^{-(r)u} c(u) du.$$

Then a decentralized economy with external addictive habits and linear technology has

i) a unique asymptotic growth rate $g = \max(\alpha^*, \Gamma)$;

ii) a unique balanced growth path

$$k(t) = k_0 e^{gt}, \quad c(t) = c_0 e^{gt}, \quad \text{and} \quad h(t) = h_0 e^{gt}$$

with either $c_0 = h_0 = (r - g)k_0$ if $g = \alpha^*$ or $c_0 = (r - g)k_0$ if $g = \Gamma$;

iii) a unique equilibrium path converging over time to the balanced growth path:

$$h(t) = \theta \hat{\varphi}_0 e^{\Gamma t} + \hat{z}_0 e^{\alpha^* t} + \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda \quad (25)$$

$$c(t) = \hat{\varphi}_0 e^{\Gamma t} (1 + \theta) + \hat{z}_0 e^{\alpha^* t} + \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda \quad (26)$$

$$k(t) = \frac{\hat{\varphi}_0 (1 + \theta)}{(r - \Gamma)} e^{\Gamma t} + \frac{\hat{z}_0}{r - \alpha^*} e^{\alpha^* t} + \int_t^\infty e^{-(r)(u-t)} \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda u} F(\lambda) d\lambda du \quad (27)$$

with

$$\frac{\hat{\varphi}_0}{(r - \Gamma)} = k(0) \left(1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)u} du \right) - \frac{h(0)}{r + \eta} + \varepsilon \frac{e^{-(r+\eta)\tau}}{r + \eta} \int_{-\tau}^0 e^{-(r)u} c(u) du \quad (28)$$

generically different from zero.

Observe that consistently with the case with τ infinite, the consumption-output ratio remains positive only if the real interest rate of the economy, $r = A - \delta$, is higher than the asymptotic growth rate, g . Also, the following remarks are useful:

Remark 6 If Condition 1 holds, the positive asymptotic growth rate is $g = \alpha^*$; on the other hand, if Condition 1 does not hold, either the economy (asymptotically) grows at the rate $g = \Gamma$ when $\Gamma > 0$, or shrinks over time to zero if $\Gamma < 0$.

Remark 7 The restriction on the initial income is less stringent than in the case with $\tau = +\infty$. In fact

$$rk_0 \geq h_0 = \varepsilon \int_{-\infty}^0 \bar{c}(u) e^{\eta u} du > \varepsilon \int_{-\tau}^0 \bar{c}(u) e^{\eta u} du$$

Remark 8 The competitive equilibrium path (25), (26), and (27) converges respectively to (30), (31), and (33) as $\tau \rightarrow \infty$.

We now investigate the nature of the convergence of the competitive equilibrium path toward the balanced growth path.

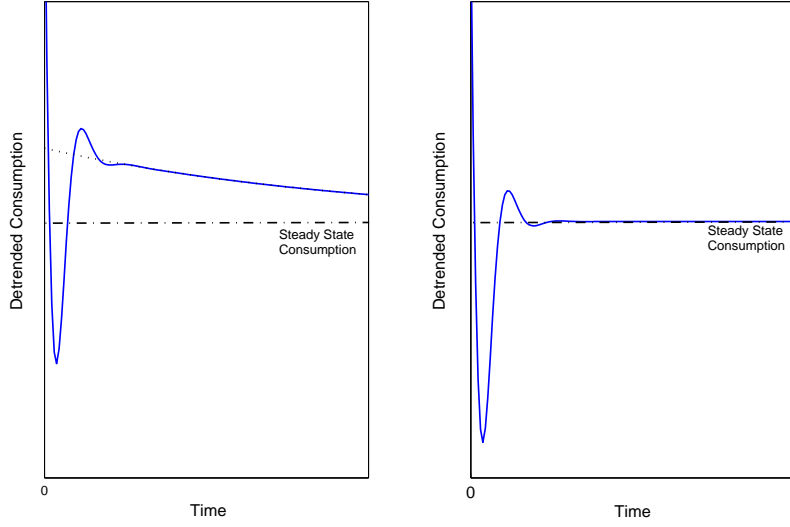


Figure 1: Monotonous (on the left) versus Oscillatory (on the right) Convergence

As observed in Remark 5, the risk adverse agents rationally choose a smooth path of $c(t) - h(t)$, as described by the first order condition (7), then equation (23) can be rewritten as

$$\dot{h}(t) = (\varepsilon - \eta)h(t) + \varepsilon h(t - \tau)e^{-\eta\tau} + g(t) \quad (29)$$

with $g(t) = \varepsilon\varphi_0 (1 - e^{-(\Gamma+\eta)\tau}) e^{\Gamma t}$. Then the habits evolution is described by a delay differential equation with forcing term $g(t)$ and for this reason their dynamics can be oscillatory as soon as a finite lag structure is chosen.¹⁴

In order to explain the different types of oscillations which may arise, we propose the following definition:

Definition 3 (Oscillatory versus monotonic convergence) *Oscillatory convergence means that the convergence to the balanced growth path is by damping fluctuations - Figure 1, right side; monotonous convergence means that the transitional dynamics are characterized by oscillations at the very beginning but the convergence to the BGP is monotonous - Figure 1, left side.*¹⁵

Monotonic convergence implies that a local analysis of the dynamics around a sufficiently small neighborhood of the balanced growth path reveals no endogenous fluctuations. In the next proposition we find parameters' conditions to have oscillatory convergence.

¹⁴The presence of delay differential equations was found critical in different economic frameworks, as in the previously mentioned model with vintage capital (e.g. Boucekkine et al. [9], [11], [13]) or with time to build (e.g. Bambi et al. [3]), to the raising of endogenous fluctuations.

¹⁵As it will become clear in the proof of Proposition 5, monotonous convergence emerges because the dominant root in determining the transitional dynamics is a real root, while a conjugate complex root in the case of oscillatory convergence.

Proposition 5 *A decentralized economy with a finite lag structure in the habits is characterized by transitional dynamics with oscillatory convergence if and only if*

$$-(\Gamma + \eta - \varepsilon)^2 + \varepsilon^2 e^{-2\tau(\Gamma + \eta)} > 0, \quad \text{and} \quad \Gamma < \bar{\Gamma} < \min(-\eta, \alpha^*)$$

where

$$\bar{\Gamma} = \max \left\{ \Gamma < -\eta, \cos(\tau \sqrt{-(\Gamma + \eta - \varepsilon)^2 + \varepsilon^2 e^{-2\tau(\Gamma + \eta)}}) = \frac{-e^{\tau(\Gamma + \eta)}(\Gamma + \eta - \varepsilon)}{\varepsilon} \right\}$$

An immediate consequence of this proposition is the following

Corollary 1 *A necessary condition for the economy to face oscillatory convergence to the BGP is that Condition 1 holds.*

5 Discussion of the key findings

5.1 Endogenous fluctuations

We start providing an intuitive explanation of the case with oscillatory convergence. For this purpose, we rewrite the habit equation when τ is finite as:

$$\dot{h}(t) = \varepsilon(c(t) - c(t - \tau)e^{-\eta\tau}) - \eta h(t)$$

and we assume that the initial history of habits is below the balanced growth path while the initial stock of capital is sufficiently high, consistently with Proposition 4. The households will decide, till the very beginning, a high consumption level because they prefer a large gap between habits and consumption at early stages then later since $c(t) - h(t) = \varphi_0 e^{\Gamma t}$ and $\Gamma < 0$. Moreover this increase in consumption will be higher than in the case with τ infinite because it has also to offset the reduction of $\dot{h}(t)$ due to the fact that consumption now plays a role in the habit formation only for a finite interval of time of amplitude τ , as expressed by the negative component $-c(t - \tau)e^{-\eta\tau}$ in equation (30). Summing up, consumption goes up and continues to increase more than in the infinite case in order to let the habits adjust quickly.

This sharp and prolonged adjustment eventually leads consumption to overshoot its trend or balanced growth path level. Observe that such consumption growth was accompanied by an accumulation of capital lower than at its trend level. For this reason, the households realize that they cannot continue to increase their consumption at that pace and decide at a date t^* to reduce it. However they cannot adjust it down too rapidly for two reasons: first because with a finite lag structure, the habits go down faster than in the case with an infinite τ as the high consumption of the past will play no role in the habit formation after a period of length τ ; secondly, because the agents' objective to reduce over

time the gap between consumption and habits must be respected. Then in the attempt to smooth the difference between consumption and habit, households will eventually arrive to consume at a rate below the trend level at a certain date t^{**} because the slow negative adjustment of consumption growth was accompanied by a slow reaction in the capital accumulation. At that point the intuition previously described can be applied again to explain the readjustment of consumption which may lead it to overshoot, again, its trend level. Oscillatory convergence is then explained by these arguments.

Therefore the source of endogenous fluctuations differs from other existing contributions in the literature which focused either on the role of the inter-temporal preference discount factor (e.g. Benhabib [5]) or on the “conflicting role” of two stocks of consumption capital, one generating addictive habit and the other one satiating behavior (see Dockner and Feichtinger [21]).

It is also worth noting that the possibility of oscillatory convergence was not found in previous contributions with consumption aspirations, because the lag parameter, τ , was always assumed to be too small (specifically equal to one) and then the consumption aspirations were growing at most at the adjusted growth rate of the shadow price of capital, in endogenous growth model, or at the exogenous growth rate of technology, in exogenous growth models (see Boldrin et al. [8]). In both cases, Condition 1 (or its equivalent in different growth models) is violated and therefore the necessary condition for the raising of oscillatory convergence is not satisfied. Therefore any local analysis around the BGP reveals monotonic convergence consistently with our result.

5.2 Consumption smoothing

Habits have been often introduced in the existing literature to increase the agents’ desire to smooth consumption. Examples in growth, international and monetary economics have been recently surveyed by Boldrin et al. [8]; for the very same reason habits have played a key role in resolving the equity premium puzzle as clearly explained by Constantinides [17]: “Habit persistence smooths consumption growth over and above the smoothing implied by the life cycle permanent income hypothesis with time separable utility. (...) This illustrate the key role of habit persistence in resolving the puzzle (...)”.

Therefore habits, as intended by these authors, are necessary to have a less steep equilibrium consumption path than in a model without habits. Looking at our previous results, it emerges that this happens when we have no consumption aspirations, or when consumption aspirations grow at a rate α^* lower than the modified growth rate of the shadow price of capital, Γ – i.e. Condition 1 does not hold. In both cases the growth rate of the economy and of the instantaneous utility is Γ . On the other hand, if consumption aspirations grow sufficiently fast, $\alpha^* > \Gamma$ – i.e. Condition 1 holds – then the consumption equilibrium path will be steeper than in a model without habits.

To illustrate better these two possibilities we propose the following Corollary of Proposition 1:

Corollary 2 *Consider an economy where $g = \bar{g}$, $r = \bar{r}$, $\delta = \bar{\delta}$, and $\gamma = \bar{\gamma}$, with \bar{g} , \bar{r} , $\bar{\delta}$, $\bar{\gamma}$ exogenously given constants. Then the transitional dynamics in a model with aspirations, τ infinite, and $\varepsilon - \eta = \bar{g}$ is characterized by a consumption path always steeper than in a standard AK model.*

5.3 Easterlin's paradox

From equations (25) and (26) of Proposition 4, it emerges immediately that the instantaneous utility remains constant over time, and it is equal to $\frac{(c_0 - h_0)^{1-\gamma}}{1-\gamma}$ as long as the parameters are specified so that $\Gamma = 0$. On the other hand, equation (27) in Proposition 4 implies a positive and time-varying growth rate of capital and output which asymptotically converges to $g = \alpha^* > 0$ if Condition 1 holds. Observe that the growth rate of utility and of output are indeed different because pinned respectively down by the (different) parameters (r, ρ, γ) and (ε, η) .

Therefore we may easily choose the parameters in order to have a global dynamics consistent with the Easterlin's paradox according to which economic growth does not raise wellbeing. This hypothesis has been tested empirically by Blanchflower and Oswald [7] among others. In these contributions, the authors test the *Easterlin's paradox*, and confirm it in several developed countries, among them U.S and U.K., as well as over different time intervals. Blanchflower and Oswald observe also that comparisons with others is one of the four key factors influencing their measure of utility and then their findings. The role of the external habits in potentially reducing the wellbeing has been empirically investigated by Luttmer [25] who confirmed that individuals' happiness is negatively affected by their position relatively to people living in the same area.

Finally it is worth noting that Condition 1, which is necessary to match the Easterlin's paradox, implies a steeper consumption path then in the same model without habits. Therefore it is not possible to use the model to explain this fact and at the same time other empirical evidences, mentioned in the previous sections, which require a parameters' specification such that the consumption equilibrium path results flatter than in the case without habits.

6 Conclusion

In this paper, we have fully analyzed the global dynamics of an AK model with external addictive habits. Habits, as defined in our contribution, may take the form of consumption aspirations and may be formed over a finite time-period. These two features may be determinant for the raising of endogenous fluctuations. We have also shown how these two features may lead to conclusion on the steepness of the equilibrium consumption path quite different with respect to the usual results in the literature, and how they may be crucial in replicating the so called Easterlin's paradox.

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Appendix

Proof of Proposition 1. Solving the system (5), (6), and (7) when τ is infinite leads to:

$$h(t) = \frac{\varepsilon\varphi_0}{\Gamma - \varepsilon + \eta} e^{\Gamma t} + \left(h_0 + \frac{\varepsilon\varphi_0}{\varepsilon - \eta - \Gamma} \right) e^{(\varepsilon - \eta)t} \quad (30)$$

$$c(t) = e^{(\varepsilon - \eta)t} \left(h_0 + \frac{\varepsilon\varphi_0}{\varepsilon - \eta - \Gamma} \right) + \varphi_0 e^{\Gamma t} \left(\frac{\eta + \Gamma}{\Gamma - \varepsilon + \eta} \right) \quad (31)$$

$$k(t) = \left[k_0 - \frac{h_0}{r - \varepsilon + \eta} - \frac{(r + \eta)\varphi_0}{(r - \varepsilon + \eta)(r - \Gamma)} \right] e^{(r)t} \quad (32)$$

$$+ \frac{1}{r - \varepsilon + \eta} \left(h_0 + \frac{\varepsilon\varphi_0}{\varepsilon - \eta - \Gamma} \right) e^{(\varepsilon - \eta)t} - \frac{(\eta + \Gamma)\varphi_0}{(\varepsilon - \eta - \Gamma)(r - \Gamma)} e^{\Gamma t} \quad (33)$$

The value of φ_0 which satisfies the transversality condition can be easily computed by looking at equation (33):

$$\tilde{\varphi}_0 = \frac{(\varepsilon - \eta - A + \delta)(\Gamma - A + \delta)}{(r + \eta)} \left(k_0 + h_0 \frac{1}{\varepsilon - \eta - A + \delta} \right) \quad (34)$$

and then consumption and capital grow asymptotically at the same rate $g = \max(\varepsilon - \eta, \Gamma)$. Moreover dividing both sides of the capital accumulation equation by $k(t)$, and taking the limit $t \rightarrow \infty$, it follows that a positive consumption over capital ratio in the long run is guaranteed when

$$\lim_{t \rightarrow \infty} \frac{c(t)}{k(t)} = r - \max(\varepsilon - \eta, \Gamma) > 0.$$

Let us now assume that $A \geq \hat{A}$ and $g < r$. We want now to verify that the conditions for an interior solution are satisfied. Let us check $c(t) > h(t)$; according to equation (3), this condition is satisfied when $\tilde{\varphi}_0 > 0$, which happens if and only if $k_0 > h_0 \frac{1}{r - \varepsilon + \eta}$.

From now on we assume that this condition is verified and we also denote with $g_1 = \min(\varepsilon - \eta, \Gamma)$.

A second condition to check is $h(t) > 0$; taking into account the definition of $h(t)$ and relation (3) we have that

$$h(t)e^{-g_1 t} = \frac{\varepsilon\varphi_0}{\Gamma - \varepsilon + \eta} e^{(\Gamma - g_1)t} + \left(h_0 + \frac{\varepsilon\varphi_0}{\varepsilon - \eta - \Gamma} \right) e^{(\varepsilon - \eta - g_1)t}$$

Thus $h(t)e^{-g_1 t}$ is an increasing function of t thus $h(t)e^{-g_1 t} > h_0 > 0$.

The last condition to check is $k(t) > 0$. Rewriting

$$k(t) = \left(h_0 + \frac{\varepsilon\tilde{\varphi}_0}{\varepsilon - \eta - \Gamma} \right) \frac{e^{(\varepsilon - \eta)t}}{r - \varepsilon + \eta} + \tilde{\varphi}_0 \left(\frac{\eta + \Gamma}{(\varepsilon - \eta - \Gamma)(\Gamma - A + \delta)} \right) e^{\Gamma t}$$

leads to prove that $k(t)e^{-g_1 t}$ is an increasing function of t , thus $k(t)e^{-g_1 t} > k_0 > 0$. It follows also immediately that the policy function is

$$\frac{c(t) - h(t)}{r - \Gamma} = \left[\frac{r + \eta - \varepsilon}{r + \eta} \right] \left[k(t) - \frac{h(t)}{r + \eta - \varepsilon} \right]$$

which is exactly the same policy function found in Constantinides [17] for the case with internal addictive habits. Then the two specification are *de facto* equivalent as also shown in Gomez

[22]. Finally the discussion on the corner solution can be found later in this Appendix for the “general” model.

■

Lemma A *The algebraic equation (6) can be rewritten as equation (16) once the new variable $z(t)$, as defined in (15), is introduced.*

Proof. Let $z(t) = h(t) - e^{\Gamma t} \varphi_0 \theta$. According to equation (6), $h(t) = \varepsilon \int_{t-\tau}^t \varphi_0 e^{\Gamma u} e^{\eta(u-t)} du + \varepsilon \int_{t-\tau}^t h(u) e^{\eta(u-t)} du$, and then we have that

$$h(t) = \varepsilon \int_{t-\tau}^t \varphi_0 e^{\Gamma u} e^{\eta(u-t)} du + \varepsilon \int_{t-\tau}^t (z(u) + e^{\Gamma u} \varphi_0 \theta) e^{\eta(u-t)} du$$

Choosing θ such that $\varepsilon e^{\Gamma t} \int_{t-\tau}^t \varphi_0 e^{(\eta+\Gamma)(u-t)} du + \varepsilon e^{\Gamma t} \int_{t-\tau}^t t \varphi_0 \theta e^{(\eta+\Gamma)(u-t)} du = e^{\Gamma t} \varphi_0 \theta$ leads to the result. ■

Proof of Lemma 1. Since $\Delta'(\lambda) = \varepsilon \int_{-\tau}^0 u e^{(\lambda+\eta)u} du < 0$, $\lim_{\lambda \rightarrow -\infty} \Delta(\lambda) = \infty$, and $\lim_{\lambda \rightarrow \infty} \Delta(\lambda) = -1$ then a unique real root, α^* , always exists, and its sign is given by the sign of $\Delta(0) = -1 + \frac{\varepsilon}{\eta}(1 - e^{-\eta\tau})$. Then after some algebra, it can be seen that $\alpha^* > 0$ if and only if $\varepsilon > \eta$ and $\tau > \tau^*$ with $\tau^* = -\frac{\log(1-\frac{\eta}{\varepsilon})}{\eta}$. If $\Gamma > 0$, $\alpha^* > \Gamma$ if and only if $\Delta(\Gamma) > 0$. This condition gives, after some algebra similar to the one just done previously the second part of the condition. We now prove that the real root is the leading root, namely, for any other complex root $\lambda = p + iq$ of $\Delta(\lambda) = 0$, then $\alpha^* > p$. To prove it, let us assume that there exist a complex root for which $p > \alpha^*$. It solves

$$1 = \varepsilon \left| \int_{-\tau}^0 e^{(\lambda+\eta)u} du \right|$$

Thus

$$1 < \varepsilon \int_{-\tau}^0 e^{(p+\eta)u} du$$

which contradicts that $\Delta(p) < 0$.

Finally the real root, α^* , solves $\varepsilon \int_{-\tau}^0 e^{(\alpha^*+\eta)u} du = 1$. Using implicit function theorem, we have

$$\frac{d\alpha^*}{d\tau} = -\frac{e^{-(\alpha^*+\eta)\tau}}{\int_{-\tau}^0 u e^{(\alpha^*+\eta)u} du} > 0$$

When τ tends to infinity, the real root solves $\varepsilon \int_{-\infty}^0 e^{(\alpha^*+\eta)u} du = 1$. This implies that $\varepsilon = \eta + \alpha^*$. It is also worth noting that as α^* is defined by $-1 + \varepsilon \int_{-\tau}^0 e^{(\alpha^*+\eta)u} du = 0$, then from the implicit functions theorems

$$\frac{d\alpha^*}{d\eta} = -1 \text{ and } \frac{d\alpha^*}{d\varepsilon} = \frac{-\int_{-\tau}^0 e^{(\alpha^*+\eta)u} du}{\varepsilon \int_{-\tau}^0 u e^{(\alpha^*+\eta)u} du}$$

We are now going to prove that there is no complex root in the strip $\Re(\lambda) \in [\min(-\eta, \alpha^*), \min(-\eta, \alpha^*)]$.

Let us now investigate the real part of the complex roots by considering the equation

$$\dot{z}(t) = \varepsilon z(t) - \varepsilon z(t-\tau) e^{-\eta\tau} - \eta z(t) \quad (35)$$

obtained by differentiating

$$z(t) = \varepsilon \int_{t-\tau}^t z(u) e^{\eta(u-t)} du$$

Let $\bar{\Delta}(\lambda) = 0$ be the characteristic equation of 35 with

$$\bar{\Delta}(\lambda) = -\lambda + (\varepsilon - \eta) - \varepsilon e^{-(\lambda+\eta)\tau}$$

The roots of $\bar{\Delta}(\lambda) = 0$ coincide with those of $\Delta(\lambda) = 0$ plus the root $\lambda = -\eta$. Then $\bar{\Delta}(\lambda) = 0$ has two real roots α^* and $-\eta$. Moreover $\bar{\Delta}^{(2)}(\lambda) < 0$, and $\bar{\Delta}(p) > 0$ for $p \in [\min(-\eta, \alpha^*), \min(-\eta, \alpha^*)]$. As we have already proved that there is no complex root with real part greater than α^* , the only case of interest is the case $-\eta < \alpha^*$. This implies that $\bar{\Delta}'(\lambda) = -1 + \varepsilon\tau > 0$.

Let us indeed consider a generic complex root $\lambda = p + iq$, with $p > -\eta$.

$$\begin{aligned} \operatorname{Re} \bar{\Delta}(p + iq) &= -p + \varepsilon - \varepsilon e^{-(p+\eta)\tau} \cos(q\tau) - \eta \\ &\geq \kappa(p) \end{aligned}$$

where $\kappa(p) = -(p + \eta) + \varepsilon e^{-(p+\eta)\tau} (1 - \cos(q\tau))$. We want to prove that $\kappa(p) > 0$. It has the same sign as

$$\begin{aligned} \tau\kappa(p) &> -\tau(p + \eta) + e^{-(p+\eta)\tau} (1 - \cos(q\tau)) \\ &> 1 - e^{-(p+\eta)\tau} \cos(q\tau) \end{aligned}$$

If $1 > \cos(q\tau) > 0$, then $\tau\kappa(p) > 1 - \cos(q\tau) > 0$. If $1 = \cos(q\tau)$, then $\tau\kappa(p) > 1 - e^{-(p+\eta)\tau} > 0$, and if $\cos(q\tau) \leq 0$, the result is immediate. Thus all complex roots of $\bar{\Delta}$ have real part smaller than $-\eta$.

■

Proof of Proposition 2. In order to determine the growth rate and the solution of $z(t)$, we apply the Laplace transform $L(\lambda) = \int_0^\infty e^{-\lambda t} z(t) dt$, on equation $z(t) = \varepsilon \int_{t-\tau}^t z(u) e^{\eta(u-t)} du$. Thus

$$\begin{aligned} L(\lambda) &= \varepsilon \int_0^\infty e^{-\lambda t} \int_{-\tau}^0 z(u+t) e^{\eta u} du dt \\ &= \varepsilon \int_{-\tau}^0 e^{\eta u} \int_0^\infty e^{-\lambda t} z(u+t) dt du \\ &= \varepsilon \int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 e^{-\lambda t} z(t) dt + L(\lambda) \right) du \end{aligned}$$

So $L(\lambda) \left(1 - \varepsilon \int_{-\tau}^0 e^{(\eta+\lambda)u} du \right) = \varepsilon \int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 e^{-\lambda t} z(t) dt \right) du$. Using inverse Laplace transform, and the result in Lemma 1, the solution admits for representation, for $t > 0$

$$z(t) = z_0 e^{\alpha^* t} + \chi_1(t) \tag{36}$$

where $\chi_1(t) = \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} F(\lambda) d\lambda$ with $\gamma > \sup \{ \operatorname{Re}(\lambda) < \alpha^*, \det(\Delta(\lambda)) = 0 \}$, $L(\gamma) = \{ \lambda \in \mathbb{C}, \operatorname{Re}(\lambda) = \gamma \}$ and with $F(\lambda) = -\varepsilon \Delta(\lambda)^{-1} \left(\int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 (h(\mu) - e^{\Gamma\mu} \varphi_0 \theta) e^{-\lambda\mu} d\mu \right) du \right)$. Finally, z_0 can be obtained by using the Residue Formula $\operatorname{res}_{\lambda=\alpha^*} e^{\lambda t} F(\lambda) = z_0 e^{\alpha^* t}$,

$$z_0 = -\frac{\varepsilon}{\Delta'(\alpha^*)} \left(\int_{-\tau}^0 e^{(\eta+\alpha^*)u} \left(\int_u^0 (h(\mu) - e^{\Gamma\mu} \varphi_0 \theta) e^{-\alpha^* \mu} d\mu \right) du \right) \tag{37}$$

Now since the arguments of the integrals are continuous functions we may use Fubini's theorem to change the order of integration, and rewriting z_0 as follows:

$$\begin{aligned} z_0 &= -\frac{\varepsilon}{\Delta'(\alpha^*)} \left[\int_{-\tau}^0 h(\mu) e^{-\alpha^* \mu} \left(\int_{-\tau}^{\mu} e^{(\eta+\alpha^*)u} du \right) d\mu - \varphi_0 \left(\int_{-\tau}^0 e^{(\eta+\alpha^*)u} \int_u^0 e^{(\Gamma-\alpha^*)\mu} d\mu du \right) \right] \\ &= -\frac{\varepsilon}{\Delta'(\alpha^*)} \left[\frac{1}{\eta+\alpha^*} \int_{-\tau}^0 h(\mu) e^{\eta\mu} d\mu - \frac{e^{-(\eta+\alpha^*)\tau}}{\eta+\alpha^*} \int_{-\tau}^0 h(\mu) e^{-\alpha^* \mu} d\mu \right. \\ &\quad \left. - \frac{\varphi_0 \theta}{(\Gamma-\alpha^*)} \left(\frac{1-e^{-(\eta+\alpha^*)\tau}}{(\eta+\alpha^*)} - \frac{1-e^{-(\Gamma+\eta)\tau}}{(\Gamma+\eta)} \right) \right] = z_0(h_0(t), \varphi_0) \quad (39) \end{aligned}$$

As $c(t) = \varphi_0 e^{\Gamma t} (1 + \theta) + z(t)$ and as $z(t)$ is defined with equation (36) with $\lim_{t \rightarrow \infty} \chi_1(t) e^{-\alpha^* t} = 0$, then we have the stated result. ■

Proof of Proposition 3. The habits and consumption solution can be found from the solution of $z(t)$ remembering that $h(t) = z(t) + e^{\Gamma t} \varphi_0 \theta$, and $c(t) = e^{\Gamma t} \varphi_0 + h(t)$. It also follows immediately that if Condition 1 holds then $\lim_{t \rightarrow \infty} \frac{\dot{z}}{z} = \lim_{t \rightarrow \infty} \frac{\dot{h}}{h} = \lim_{t \rightarrow \infty} \frac{\dot{c}}{c} = \alpha^*$. ■

Proof of Proposition 4. By substituting the equation of consumption (20) found in proposition (3), into the capital accumulation equation and integrating it, we arrive to

$$k(t) = k_0 e^{(r)t} - \int_0^t e^{-(r)(u-t)} \left(\varphi(0) (1 + \theta) e^{\Gamma u} + z_0 e^{\alpha^* u} + \chi_1(u) \right) du$$

Given the transversality condition $\lim_{t \rightarrow \infty} k(t) e^{-(r)t} = 0$, it must be that

$$k_0 = \varphi_0 (1 + \theta) \frac{1}{(r - \Gamma)} + \frac{z_0}{r - \alpha^*} + \int_0^\infty e^{-(r)u} \chi_1(u) du \quad (40)$$

where z_0 and $\chi_1(u)$ have been defined in proposition (3) as

$$z_0 = -\frac{\varepsilon}{\Delta'(\alpha^*)} \left(\int_{-\tau}^0 e^{(\eta+\alpha^*)u} \left(\int_u^0 (h(\mu) - e^{\Gamma\mu} \varphi_0 \theta) e^{-\alpha^* \mu} d\mu \right) du \right)$$

and

$$\chi_1(t) = \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda t} \left(-\varepsilon \Delta(\lambda)^{-1} \left(\int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 (h(\mu) - e^{\Gamma\mu} \varphi_0 \theta) e^{-\lambda\mu} d\mu \right) du \right) \right) d\lambda$$

with $\gamma > \sup \{ \text{Re}(\lambda) < \alpha^*, \det(\Delta(\lambda)) = 0 \}$, $L(\gamma) = \{ \lambda \in \mathbb{C}, \text{Re}(\lambda) = \gamma \}$. Since $r > \alpha^*$, otherwise in the long run the positive sign of the consumption-output ratio is not guaranteed, then $\int_0^\infty e^{-(r)u} \chi_1(u) du$ exists.

Equation (40) leads to a linear equation in φ_0 , or equivalently in c_0 as $c_0 = \varphi_0 + h_0$. After some calculus we may find the value of $\varphi_0 = \hat{\varphi}_0$ which make the transversality condition hold

$$\hat{\varphi}_0 = \frac{\left(k_0 + \frac{\frac{\varepsilon}{\Delta'(\alpha^*)} \int_{-\tau}^0 e^{(\eta+\alpha^*)u} \left(\int_u^0 h(\mu) e^{-\alpha^* \mu} d\mu \right) du}{r - \alpha^*} \right)}{\left[(1 + \theta) \frac{1}{(r - \Gamma)} + \frac{\frac{\varepsilon}{\Delta'(\alpha^*)} \left(\int_{-\tau}^0 e^{(\eta+\alpha^*)u} \left(\int_u^0 e^{\Gamma\mu} \theta e^{-\alpha^* \mu} d\mu \right) du \right)}{r - \alpha^*} \right.} \quad (41)$$

$$\left. + \int_0^\infty e^{-(r)v} \frac{1}{2i\pi} \int_{L(\gamma)} e^{\lambda v} \varepsilon \Delta(\lambda)^{-1} \left(\int_{-\tau}^0 e^{(\eta+\lambda)u} \left(\int_u^0 e^{\Gamma\mu} \theta e^{-\lambda\mu} d\mu \right) du \right) d\lambda dv \right]$$

Observe that $\hat{\varphi}_0$ depends only on the initial conditions, k_0 and $h_0(\cdot)$, namely $\hat{\varphi}_0 = \hat{\varphi}_0(k_0, h_0(\cdot))$.

In order to have simpler expression for φ_0 , let us compute $c-h$. From the first order condition we have

$$c(t) - h(t) = \hat{\varphi}_0 e^{\Gamma t} \quad (42)$$

On the other hand, using equation (30) and (32) we have

$$\begin{aligned} k(t) - \int_t^\infty e^{-(r)(u-t)} h(u) du &= k(t) - \int_t^\infty e^{-(r)(u-t)} \left(\theta \hat{\varphi}_0 e^{\Gamma u} + \hat{z}_0 e^{\Gamma \alpha^* u} + \chi_1(u) \right) du \\ &= k(t) - \frac{\theta \hat{\varphi}_0 e^{\Gamma t}}{r - \Gamma} - \frac{\hat{z}_0 e^{\alpha^* t}}{r - \alpha^*} - \int_t^\infty e^{-(r)(u-t)} \chi_1(u) du \\ &= \frac{\hat{\varphi}_0 e^{\Gamma t}}{r - \Gamma} \end{aligned}$$

Thus

$$\frac{c-h}{r-\Gamma} = k(t) - \int_t^\infty e^{-(r)(u-t)} h(u) du \quad (43)$$

Moreover, we have:

$$\dot{h} = \varepsilon (c(t) - c(t-\tau) e^{-\eta \tau}) - \eta h(t)$$

then part integration of $\int_t^\infty e^{-(r)(u-t)} h(u) du$ yields to

$$\begin{aligned} \int_t^\infty e^{-(r)(u-t)} h(u) du &= e^{(r)t} \int_t^\infty e^{-(r)u} h(u) du \\ &= \frac{h(t)}{r} + \int_t^\infty \frac{e^{-(r)(u-t)} (\varepsilon (c(u) - c(u-\tau) e^{-\eta \tau}) - \eta h(u))}{r} du \\ &= \frac{h(t)}{r} + \frac{\varepsilon}{r} \int_t^\infty e^{-(r)(u-t)} c(u) du \\ &\quad - \frac{\varepsilon e^{-\eta \tau}}{r} \int_t^\infty e^{-(r)(u-t)} c(u-\tau) du \\ &\quad - \frac{\eta}{r} \int_t^\infty e^{-(r)(u-t)} h(u) du \end{aligned}$$

which rewrites

$$\begin{aligned} \int_t^\infty e^{-(r)(u-t)} h(u) du \left(1 + \frac{\eta}{r} \right) &= \frac{h(t)}{r} + \frac{\varepsilon}{r} k(t) - \frac{\varepsilon e^{-\eta \tau} e^{-(r)\tau}}{r} \int_{t-\tau}^\infty e^{-(r)(u-t)} c(u) du \\ &= \frac{h(t)}{r} + \frac{\varepsilon}{r} k(t) - \frac{\varepsilon e^{-\eta \tau} e^{-(r)\tau}}{r} \int_{t-\tau}^t e^{-(r)(u-t)} c(u) du \\ &\quad - \frac{\varepsilon e^{-\eta \tau} e^{-(r)\tau}}{r} \int_t^\infty e^{-(r)(u-t)} c(u) du \end{aligned}$$

Integrating equation (16) yields

$$\int_t^\infty e^{-(r)(u-t)} c(u) du = k(t)$$

Thus going back to equation (43) we obtain the following policy function

$$\frac{c-h}{r-\Gamma} = k - \frac{1}{r+\eta} \left[h(t) + \varepsilon (1 - e^{-(r+\eta)\tau}) k(t) - \varepsilon e^{-\eta \tau} e^{-(r)\tau} \int_{t-\tau}^t e^{-(r)(u-t)} c(u) du \right] \quad (44)$$

Evaluating the previous expression at $t = 0$ and using equation 42, we thus deduce that

$$\widehat{\varphi}_0 = (r - \Gamma) \left(k(0) \left(1 - \varepsilon \int_{-\tau}^0 e^{(r+\eta)u} du \right) - \frac{h(0)}{r + \eta} + \varepsilon \frac{e^{-(r+\eta)\tau}}{r + \eta} \int_{-\tau}^0 e^{-(r)u} c(u) du \right)$$

Moreover, initial conditions such that $\widehat{\varphi}_0 > 0$ yield to interior solutions as $c - h = \widehat{\varphi}_0 e^{\Gamma t} > 0$. Condition on $h(t) > 0$, comes from the definition of h .

$$\begin{aligned} h(t) &= \varepsilon \int_{t-\tau}^t \left(\widehat{\varphi}_0 e^{\Gamma u} + h(u) e^{\eta(u-t)} \right) du \\ &= \varepsilon \widehat{\varphi}_0 e^{\Gamma t} \int_{-\tau}^0 e^{(\eta+\Gamma)u} du + \varepsilon \int_{t-\tau}^t h(u) e^{\eta(u-t)} du \end{aligned}$$

Thus if $h(u) > 0$ on $[0, t]$, thus $h(t) > 0$.

Condition $k(t) > 0$ can be proved using

$$k(t) e^{-(r)t} = \int_t^\infty \left(\widehat{\varphi}_0 e^{\Gamma u} + h(u) e^{\eta(u-t)} \right) du > 0.$$

■

Proof of Proposition 5. In this Proposition, we are looking for conditions such that there exist complex roots with real part in the strip $\Re(\lambda) \in [\min(\alpha^*, \Gamma), \max(\alpha^*, \Gamma)]$. In lemma 1, we have prove that characteristic equation $\Delta(\lambda) = 0$ has no complex root with real part greater than $\min(\alpha^*, -\eta)$. Thus a necessary condition to have oscillatory convergence is to consider $\Gamma < -\eta < 0$. Thus in the following, we would restrict to the case $\Gamma < -\eta < 0$.

To prove damping fluctuations around the balanced growth path (i.e. oscillatory convergence), we need to prove that complex roots exist in the strip $]\Gamma, \alpha^*]$.

We are going to prove that there exists $\bar{\Gamma} < -\eta$ such that if $\Gamma > \bar{\Gamma}$ convergence is monotonous and if $\Gamma < \bar{\Gamma}$ then it is oscillatory. To determine $\bar{\Gamma}$ we need to look at the smallest value of Γ such as there exist complex roots of $\Delta(\lambda) = 0$ with real part Γ . Let us consider $\lambda = \Gamma + iq$ to be such a root. Splitting real and complex part of $\Delta(\lambda) = 0$ yields to solve the two following equations

$$\begin{cases} e^{\tau(\Gamma+\eta)} \left((\Gamma + \eta)^2 + q^2 - \varepsilon(\Gamma + \eta) \right) = -\varepsilon(\Gamma + \eta) \cos(q\tau) + \varepsilon q \sin(q\tau) \\ e^{\tau(\Gamma+\eta)} q \varepsilon = \varepsilon(\Gamma + \eta) \sin(q\tau) + \varepsilon q \cos(q\tau) \end{cases}$$

squaring both sides of these two equations and adding them together yields to

$$e^{2\tau(\Gamma+\eta)} \left(\left((\Gamma + \eta)^2 + q^2 - \varepsilon(\Gamma + \eta) \right)^2 + \varepsilon^2 q^2 \right) = \varepsilon^2 \left[(\Gamma + \eta)^2 + q^2 \right]$$

If $-(\Gamma + \eta - \varepsilon)^2 + \varepsilon^2 e^{-2\tau(\Gamma+\eta)} > 0$, there exists positive real roots q of the previous equation and they solve

$$q = \sqrt{-(\Gamma + \eta - \varepsilon)^2 + \varepsilon^2 e^{-2\tau(\Gamma+\eta)}}$$

As we would like to stress the dependency on Γ , we would denote q as $q(\Gamma)$. Moreover, equation 6 implies that

$$\cos(q(\Gamma)\tau) = \varphi(\Gamma) \tag{45}$$

where

$$\varphi(\Gamma) = \frac{-e^{\tau(\Gamma+\eta)}(\Gamma + \eta - \varepsilon)}{\varepsilon}$$

As $\Gamma \rightarrow \cos(q(\Gamma)\tau)$ oscillates between -1 and 1 as Γ varies from $-\infty$ to $-\eta$ while $\varphi(\Gamma)$ varies from 1 to 0^- . Thus it exists $\bar{\Gamma}$ which solves (45) and then $\Delta(\lambda) = 0$ has complex roots with real part Γ . Let us define

$$\bar{\Gamma} = \max\{\Gamma < -\eta, \cos(q(\Gamma)\tau) = \varphi(\Gamma)\}.$$

This piece of information allows us to identify the set of parameters which leads to oscillatory convergence.

■

Proof of Corollary 2. The strategy of the proof is to show that detrended consumption in the economy with aspirations is less smooth than in the standard AK model by showing that $c_0 - c_0^{AK} < 0$ and $c_\infty - c_\infty^{AK} > 0$ where the variables with the index AK refer to the standard AK model without habits, while the others to the model with consumption aspirations and $\bar{g} = \varepsilon - \eta$.

Let us begin to prove the first inequality. We know that

$$c_0^{AK} = (\bar{r} - \bar{g})k_0$$

while from Proposition 1 we may easily find that

$$c_0 = \left(1 + \frac{\Gamma - \bar{r}}{\bar{r} + \eta}\right) h_0 + \frac{(\bar{g} - \bar{r})(\Gamma - \bar{r})}{\bar{r} + \eta} k_0 \quad \text{with } h_0 < (\bar{r} - \bar{g})k_0$$

Combining the two expressions and after some algebra we have that

$$c_0 - c_0^{AK} = \frac{\Gamma + \eta}{\bar{r} + \eta} [h_0 - (\bar{g} - \bar{r})] < 0 \quad \text{always}$$

The second inequalities can be proved similarly. First in the standard AK model without habits, consumption jumps immediately to its balanced growth path and remains there forever, then $c_0^{AK} = c_\infty^{AK}$ where the latter indicates detrended consumption at $t \rightarrow \infty$. On the other hand, detrended consumption at $t \rightarrow \infty$ in the model with aspirations is equal, from Proposition 1, to

$$c_\infty = \left(1 + \frac{\varepsilon(\Gamma - \bar{r})}{(\bar{g} - \Gamma)(\bar{r} + \eta)}\right) h_0 + \frac{\varepsilon(\bar{g} - \bar{r})(\Gamma - \bar{r})}{(\bar{g} - \Gamma)(\bar{r} + \eta)} k_0$$

Then we may easily compute that

$$c_\infty - c_\infty^{AK} = \frac{(\bar{g} - \bar{r})(\Gamma + \eta)}{(\bar{g} - \Gamma)(\bar{r} + \eta)} (h_0 - (\bar{g} - \bar{r})) > 0 \quad \text{always}$$

■